
Buffon's Needle Problem

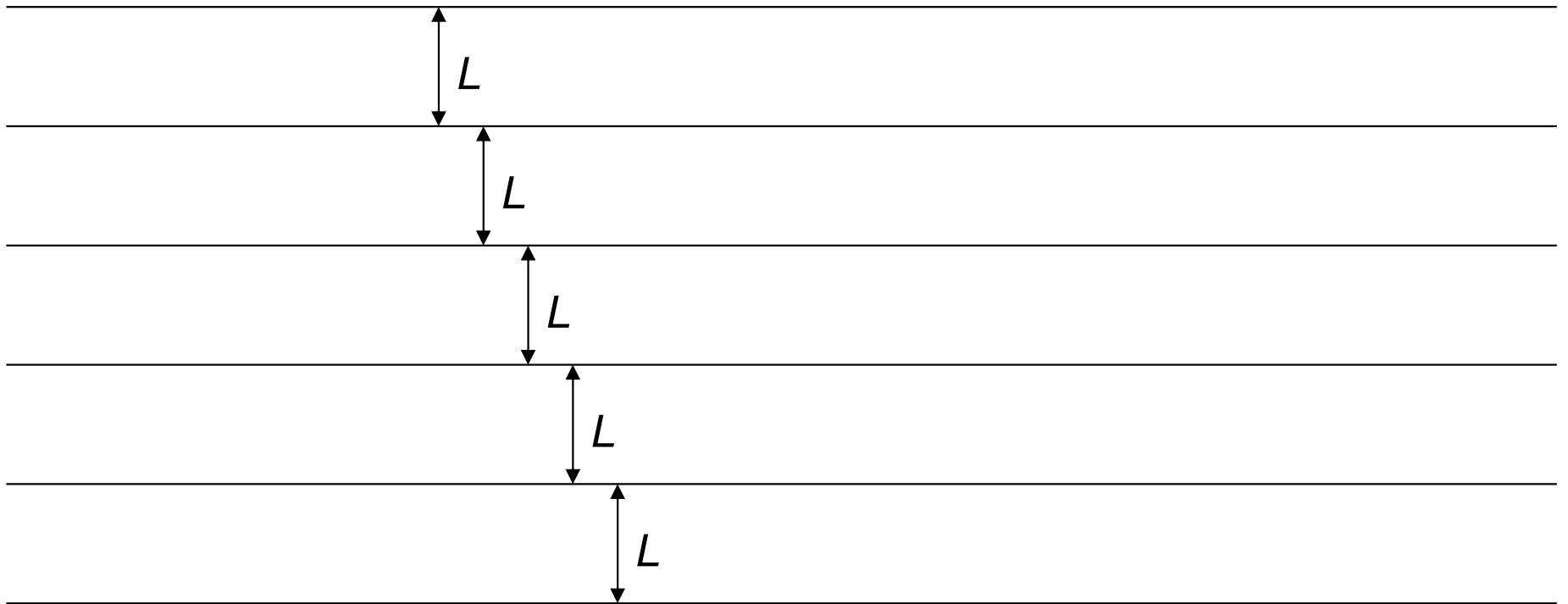
Clay Ford
October 15, 2010

Backstory

- Georges-Louis Leclerc, Comte de Buffon
 - Known primarily for his contributions to Natural history (35 volumes) in 18th century
 - Posed his needle problem in a paper published in 1777
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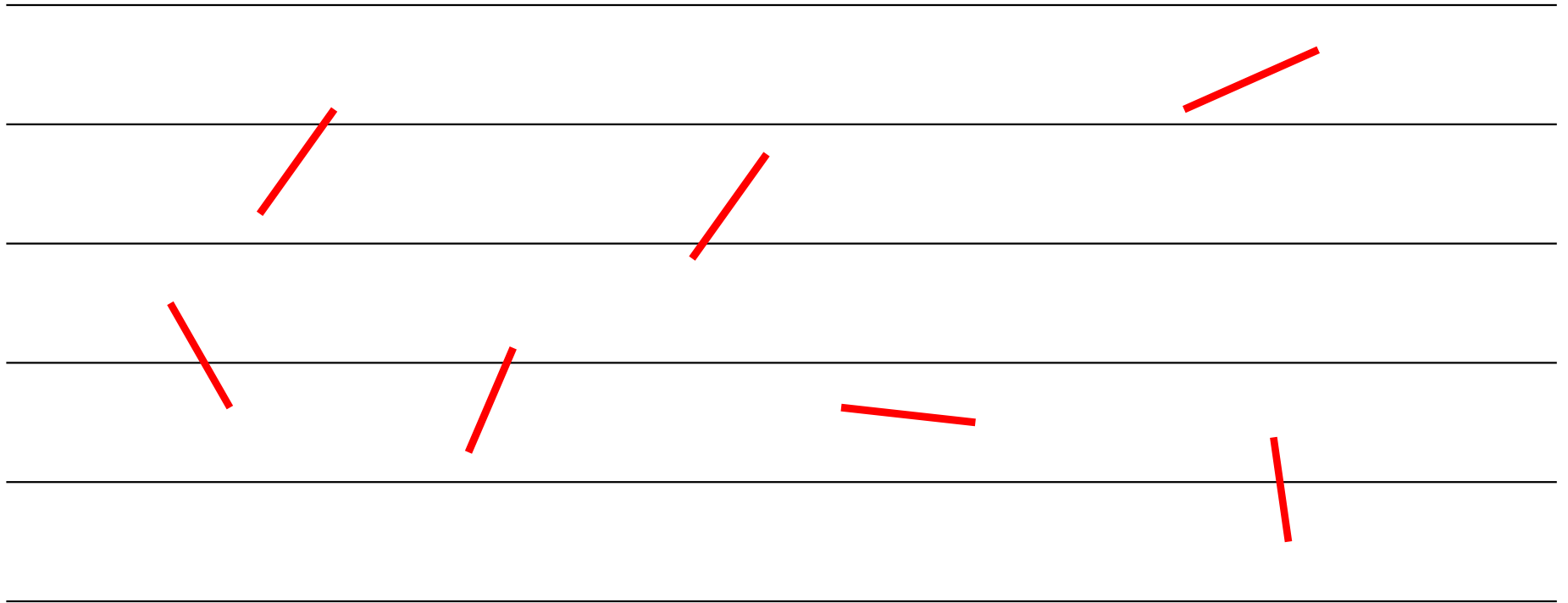
Problem Setup

- A floor with parallel lines, all an equal distance apart (call it L):



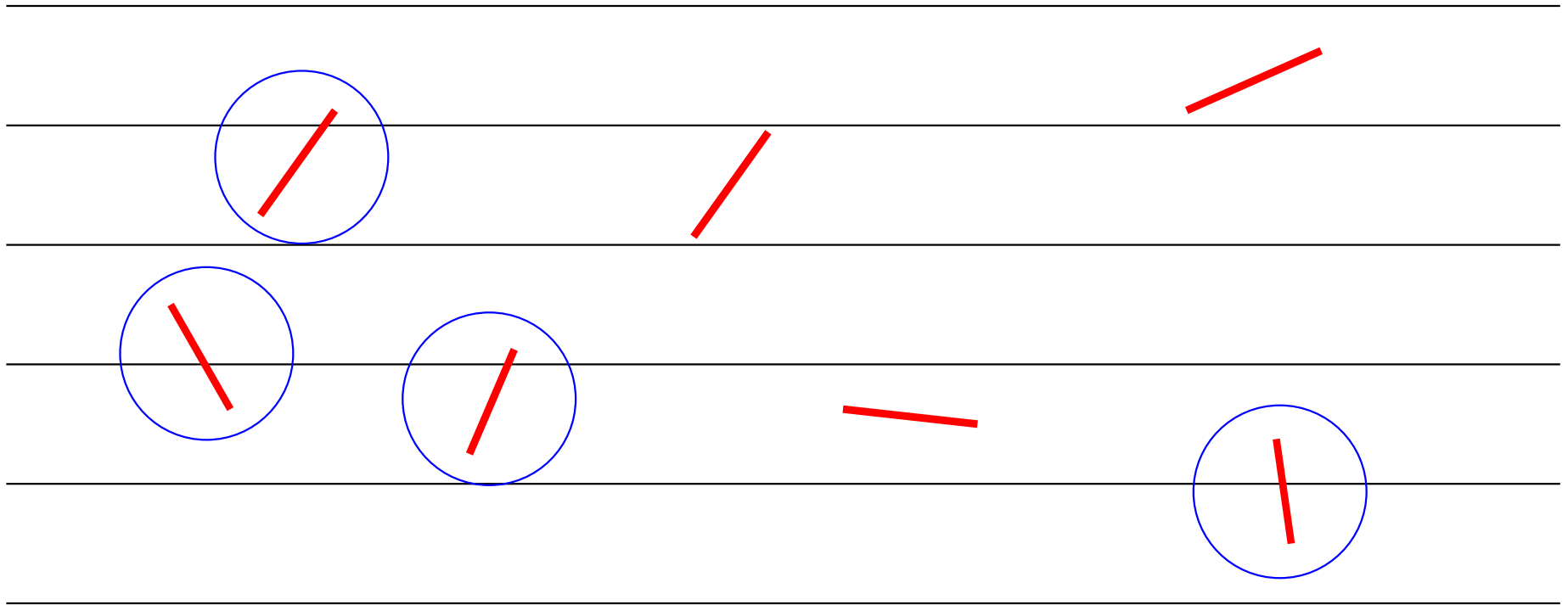
Problem Setup

- Imagine we drop some needles with length $a = L$ on the floor:



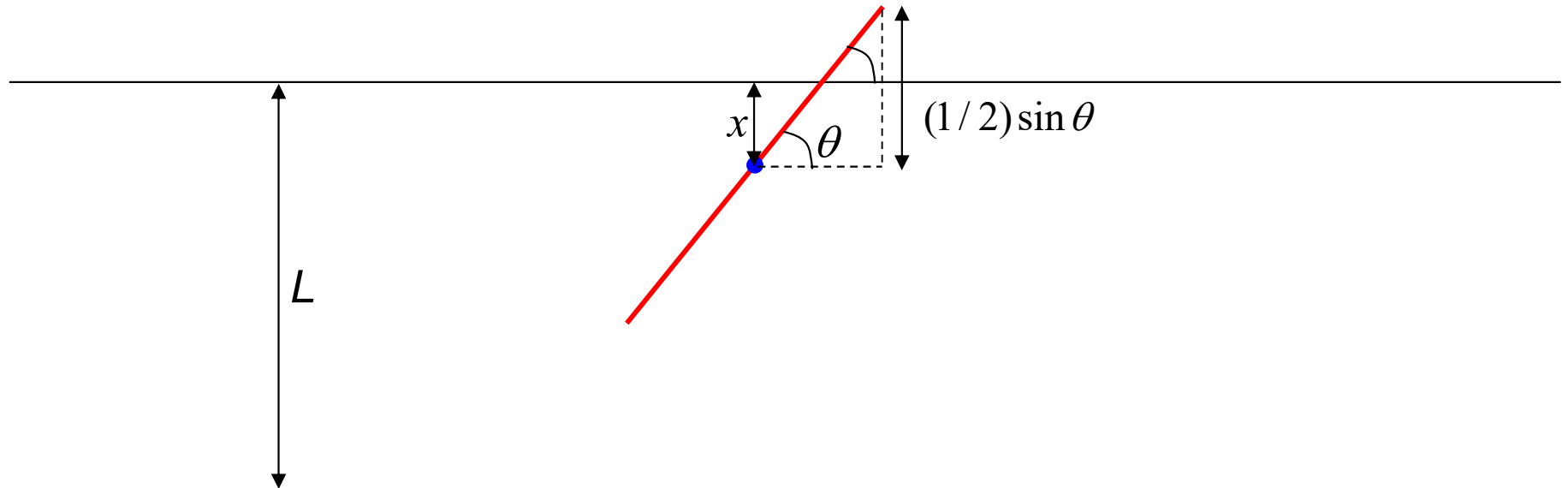
Problem Setup

- What's the probability a needle intersects a line?



When does a needle intersect a line?

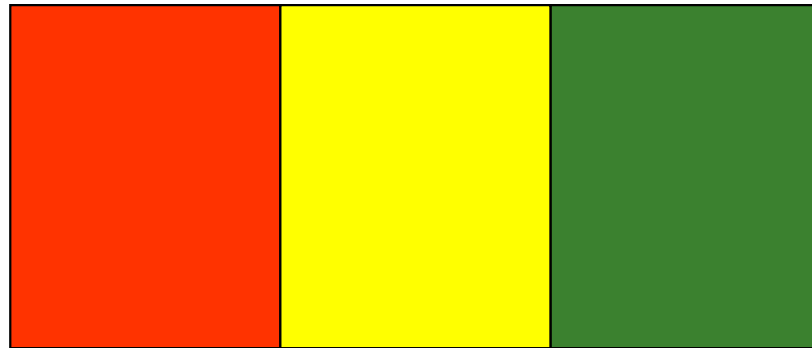
- When $x \leq (1/2)\sin \theta$, where x is distance from midpoint of needle to closest line.



For simplicity, we let $L = a = 1$

What's the probability?

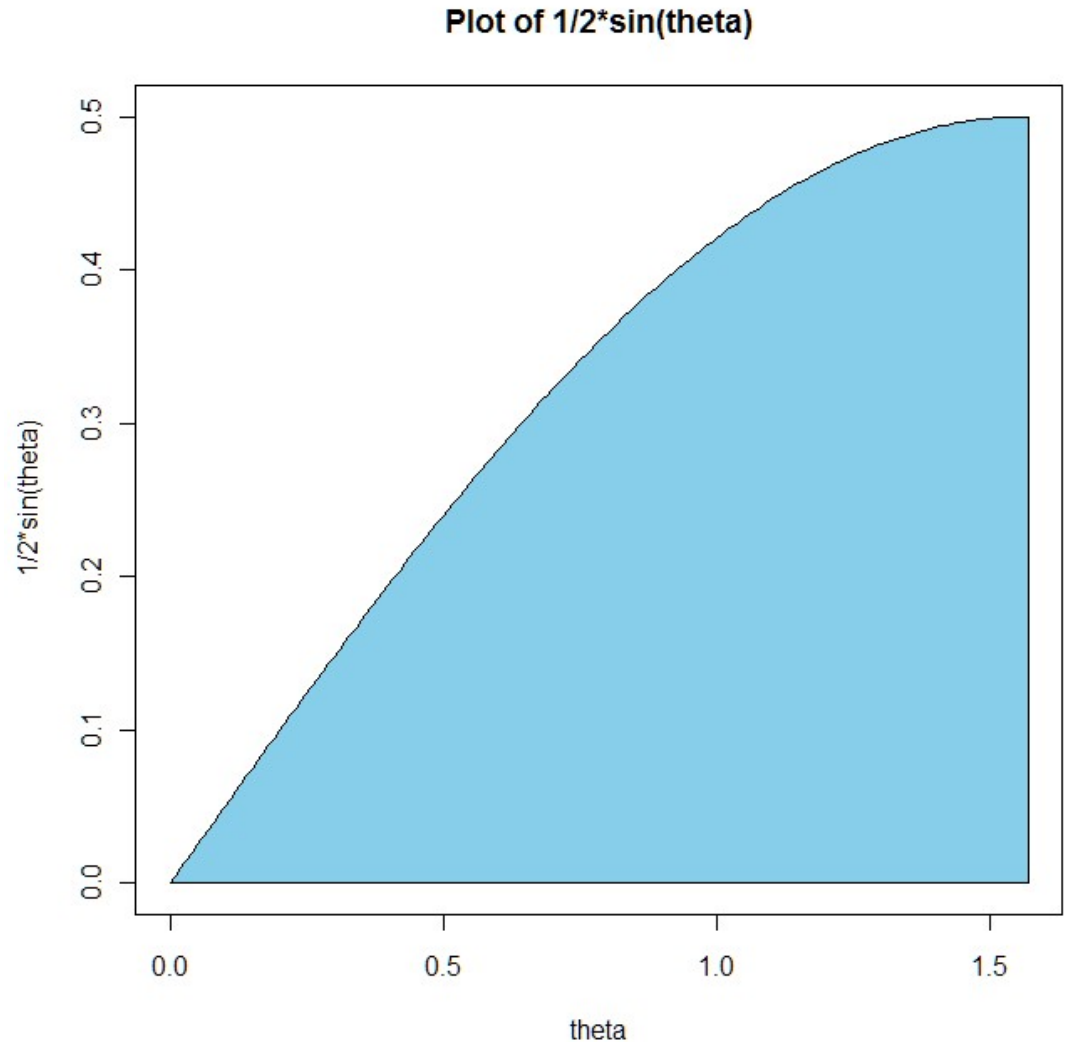
- We seek $P(x \leq (a / 2) \sin \theta)$
- Frame the problem in terms of geometry
- For example:



$$P(\text{landing on green}) = \frac{\text{green area}}{\text{total area}}$$

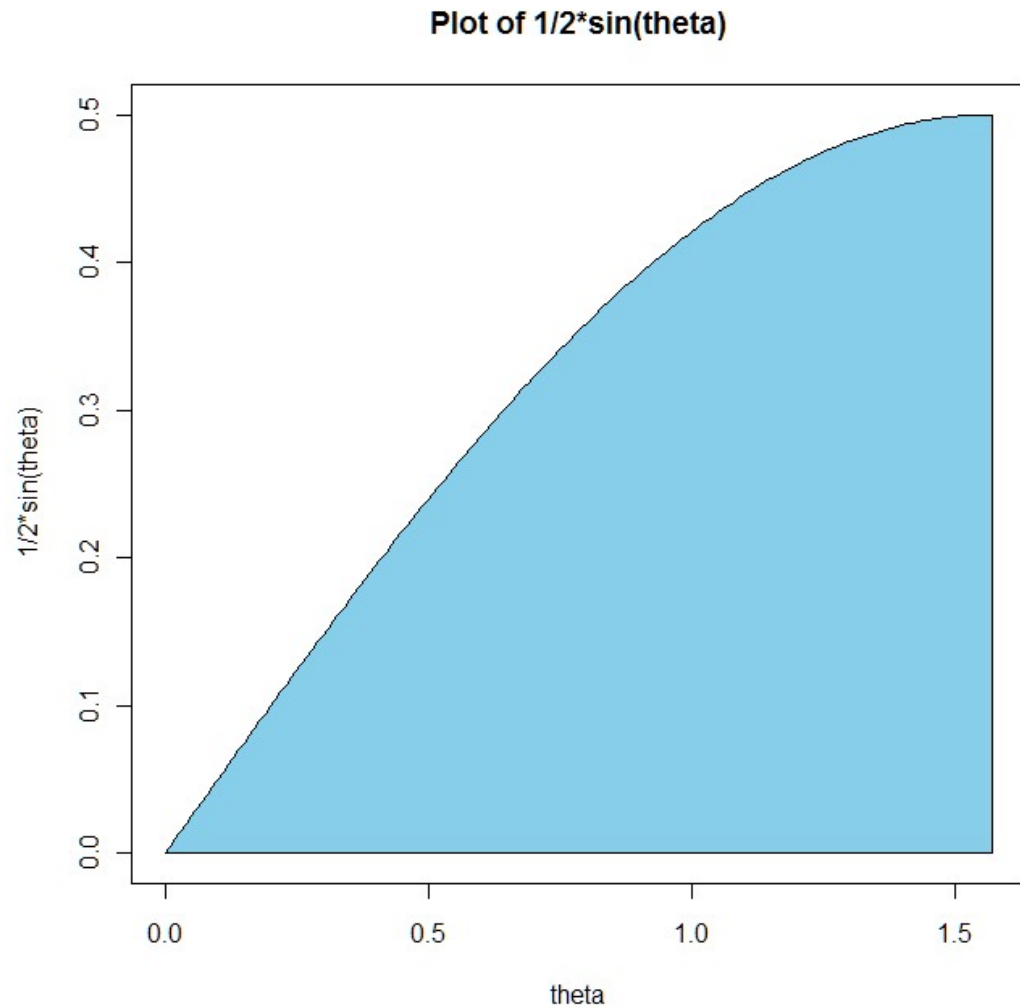
Define a region

- For simplicity, set $L = a = 1$
- Plot $(1/2)\sin\theta$ as a function of θ
- $0 \leq \theta \leq \pi/2$
- The region in blue represents when $x \leq (1/2)\sin\theta$



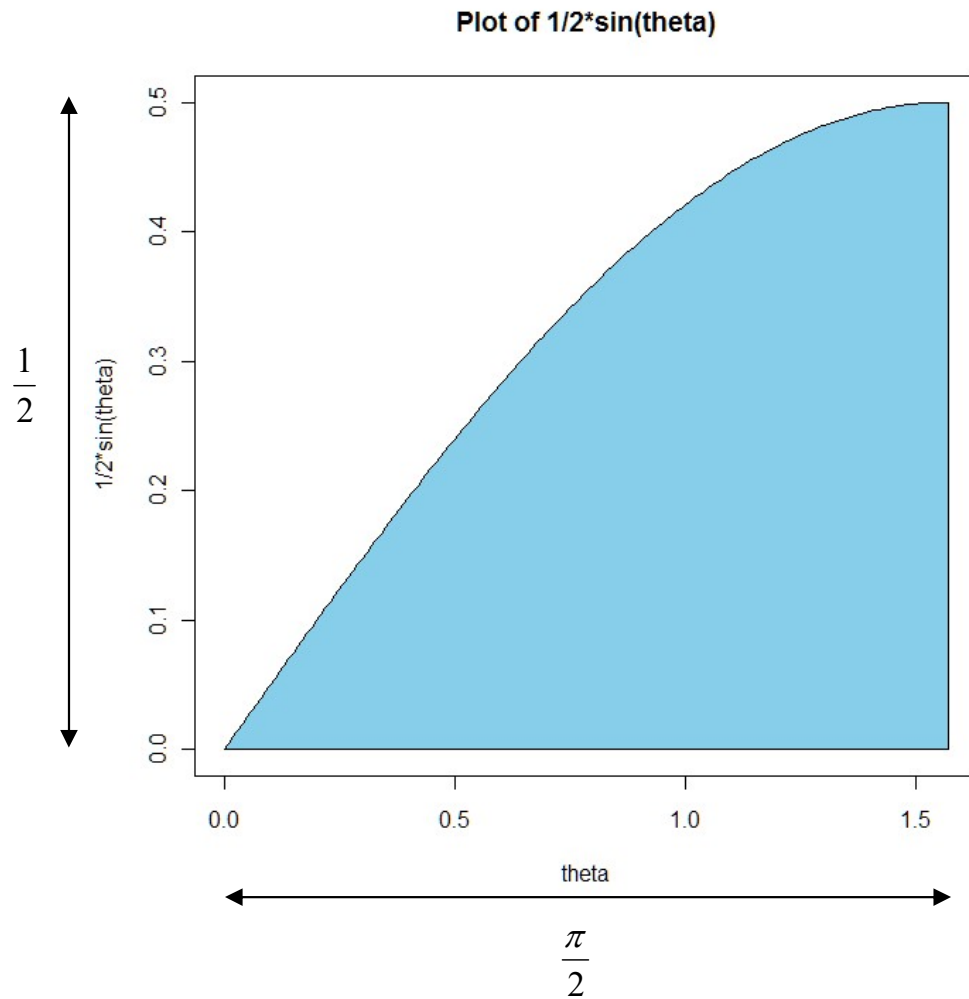
Area of blue region

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \frac{1}{2} \sin \theta \, d\theta \\ &= \frac{1}{2} \left[-\cos \frac{\pi}{2} - (-\cos 0) \right] \\ &= \frac{1}{2} \end{aligned}$$



Area of total region

$$A = \frac{1}{2} \frac{\pi}{2} = \frac{\pi}{4}$$



Now find probability

$$P(x \leq (1/2) \sin \theta)$$

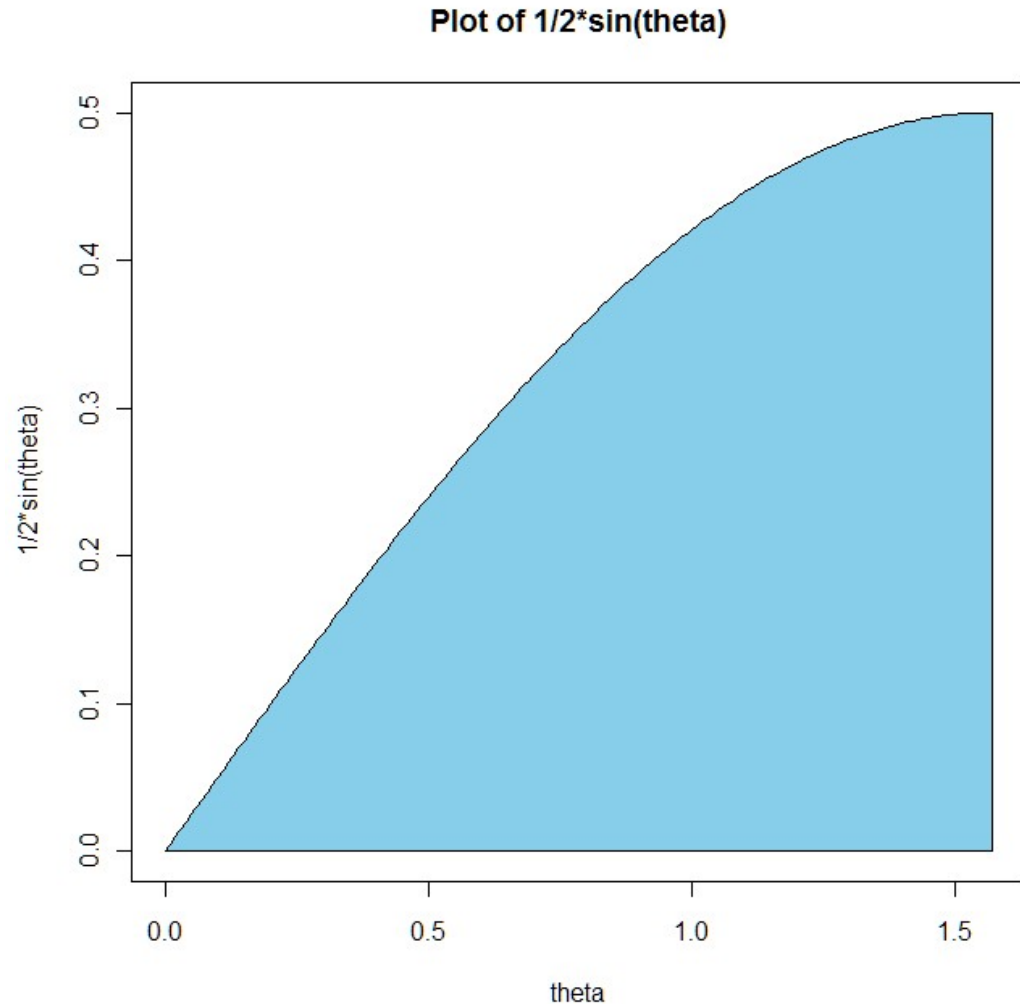
$$= \frac{\frac{1}{2}}{\frac{\pi}{4}} = \frac{1}{2} \frac{4}{\pi}$$

$$= \frac{2}{\pi} \approx 0.6366197$$

In general, we can show

$$P(x \leq (a/2) \sin \theta) = \frac{2a}{L\pi}$$

for $a \leq L$



Let's Drop Some Needles

- Again, assume $a = 1$ and $L = 1$
- $P(\text{line intersection}) = 2/\pi$
- The process of dropping n needles and observing intersections has the Binomial setting
 - Fixed number of trials
 - Independence
 - Same probability on each throw of a needle
 - Two outcomes: cross a line/do not cross a line

Expected number of crossings

- Under Binomial setting, expected number of successes (i.e., line intersections) is

$$E(X) = np$$

- Example: say we toss 1000 needles
 - Expected number of line crossings is $1000 \times 2/\pi$
 - That's about 637

Estimating probability

- Further, we could toss 1000 needles and estimate p if we didn't know it
 - Say we toss 1000 needles and observe 650 hits
 - $\hat{p} = \frac{x}{n} = \frac{650}{1000} = 0.65$

The implication

- But let's pretend we knew p but did not know the value for π
- Notice we can solve for π :

$$\hat{p} \approx p = \frac{2}{\pi} \approx \frac{650}{1000}$$

$$\Rightarrow \pi \approx \frac{2n}{\text{number of hits}} = \frac{(2)1000}{650} \approx 3.08$$

- The frequency of needles crossing lines can be used to estimate π !

Simulation

- Using R or SAS, we can create a program that simulates needle tosses and estimates π
 - randomly generate an acute angle
 - randomly generate the distance of midpoint of needle to nearest line
 - If $x \leq (1/2)\sin\theta$ then record a line crossing
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Simulation Code - R

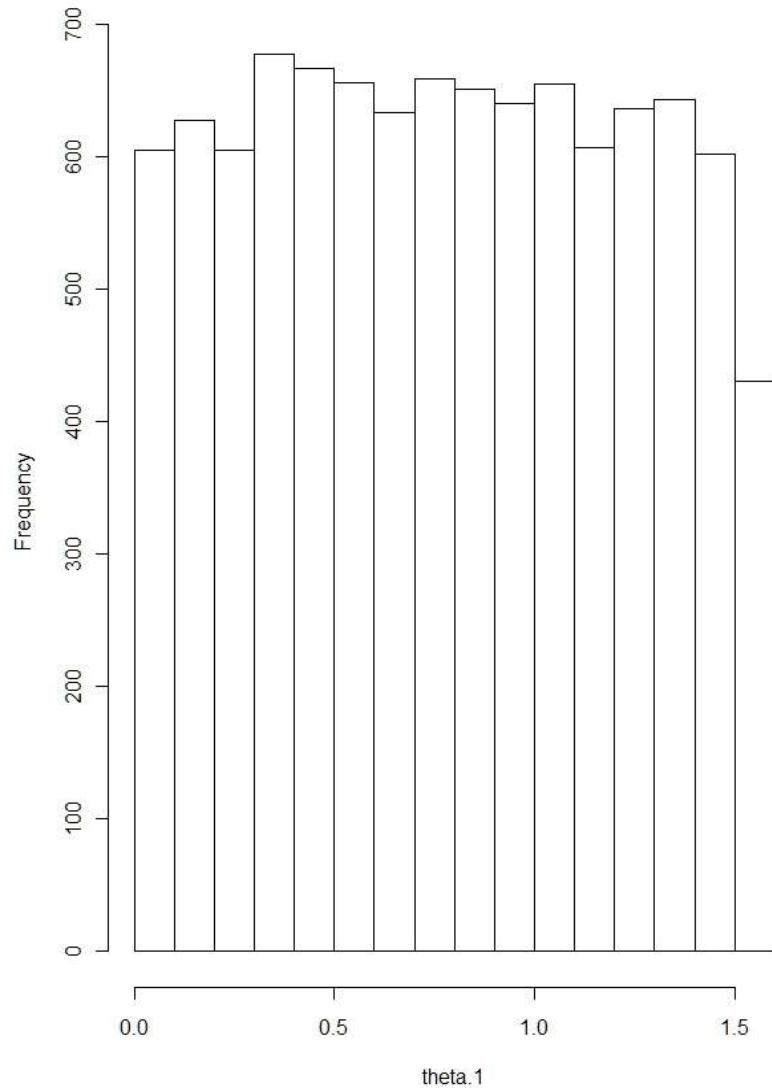
```
a <- 1 # length of needle
L <- 1 # distance between lines
n <- 100000 # number of dropped needles
hit <- 0
for(i in 1:n) {
  x <- runif(1,0,1)
  y <- runif(1,0,1)
  while(x^2 + y^2 > 1) { # no points outside of unit circle
    x <- runif(1,0,1)
    y <- runif(1,0,1)
  }
  theta <- atan(y/x) # the random angle
  d <- runif(1,0,(L/2)) # distance of needle midpoint to
                        nearest line
  if(d <= (a/2)*sin(theta)) {
    hit <- hit + 1
  }
}
pi.est <- (n*2*a)/(hit*L)
```

Simulation Code - SAS

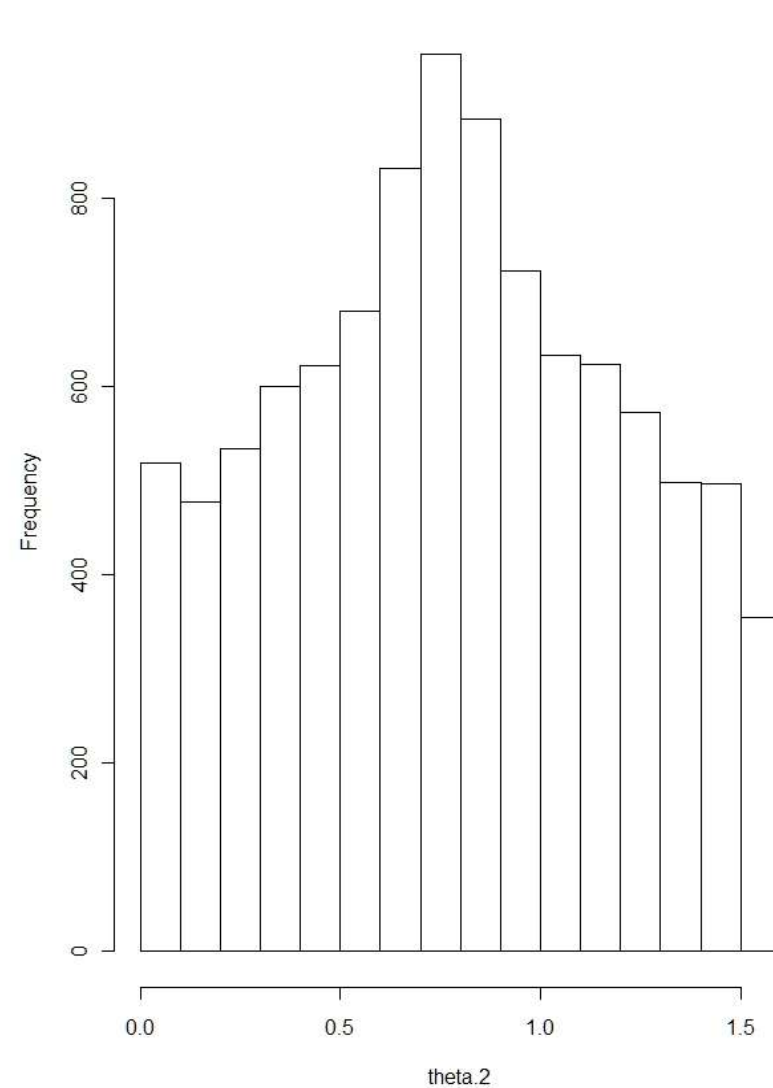
```
data _null_;  
  file log;  
  hit = 0;  
  a = 1; /*length of needle*/  
  L = 1; /*distance between lines*/  
  n = 1000000;  
  do i = 1 to n;  
    x = ranuni(0);  
    y = ranuni(0);  
    do while (x**2 + y**2 > 1);  
      x = ranuni(0);  
      y = ranuni(0);  
    end;  
    theta = atan(y/x); /*random angle*/  
    d = (a/2)*ranuni(0); /*distance from middle of needle to nearest line*/  
    if d <= (L/2)*sin(theta) then hit = hit + 1;  
  end;  
  pi_est = (n*2*a)/(hit*L);  
  put 'estimate of pi: ' pi_est;  
run;
```

Why keep points in Unit Circle?

Distribution of angles when points limited within Unit Circle



Distribution of angles when points allowed out of Unit Circle



Simulation Results ($a = L$)

- For $a = 1, L = 1, n = 10,000$
 - 3.148119
- For $a = 1, L = 1, n = 100,000$
 - 3.140605
- For $a = 1, L = 1, n = 1,000,000$
 - 3.141680

First 6 digits of pi:
3.141592

Simulation Results ($a < L$)

- For $a = 1, L = 1.25, n = 10,000$
 - 3.175863
- For $a = 1, L = 1.25, n = 100,000$
 - 3.134735
- For $a = 1, L = 1.25, n = 1,000,000$
 - 3.139847

First 6 digits of pi:
3.141592